are equivalent, as each of them is equivalent to (5).

Suppose that the greatest common divisor of the elements of the system $(m_1, m_2, \ldots, m_a)$ is unity. From

$$(m_1, m_2, \ldots, m_a) \backsim (1)$$

it follows that it is possible to find a numbers $c_1, c_2, \ldots, c_a$ such that

$$c_1 m_1 + c_2 m_2 + \ldots + c_a m_a = 1.$$

In particular, we have the fundamental theorem that it is possible to find two numbers x, y such that

$$m_1 x + m_2 y = 1,$$

whenever m_1 and m_2 are prime to each other.

GENERALIZATION OF A FUNDAMERTAL THEOREM IN THE GEOMETRY OF THE TRIANGLE.

By PROF. M. W. HASKELL.

The theorem in question is of fundamental importance in the geometry of the triangle,* and may be stated as follows:

If A', B', C' be points chosen at will on the sides BC, CA, AB of any triangle ABC, the circles $AB'C'$, $BC'A'$, $CA'B'$ pass through one and the same point O.

In a communication presented to the Chicago Section of the American Mathematical Society January 2, 1902, I extended this theorem to the tetrahedron in the following form:

Let F, G, H, P, Q, R be any points on the edges AD, BD, CD, BC, CA, AB, respectively, of any tetrahedron; the four spheres $AFQR$, $BGRP$, $CHPQ$, $DFGH$ pass through one and the same point O.

The theorem is, however, capable of generalization to space of any number of dimensions without any difficulty. I will therefore state and prove it at once for space of n dimensions,—understanding by a spherical space of three dimensions, S_3, a space every section of which by a flat space of three dimensions, R_3, is an ordinary sphere, and in general by a spherical S_{n-1} a space every section of which by a flat R_{n-1} is a spherical S_{n-2}. A spherical S_{n-1} will evidently be determined by $n+1$ points of which never more than two lie on the same line nor more than three in the same plane, etc. The general theorem may then be stated in the following words:

*McClelland, *Geometry of the Circle*, page 40; see also Rouche et de Comberousse, *Traite de Geometrie*, 7th edition, Vol. I, page 486.

Let A_1, A_2, $A_3 \ldots . A_{n+1}$ be the vertices of an $(n+1)$-point in a flat space of n dimensions, and select at will on each edge A_iA_k of this $(n+1)$-point a point A_{ik}. The $n+1$ spherical S_{n-1} determined by the groups of points such as $[A_i;\ A_{i1}, A_{i2}, \ldots . A_{i,\,n+1}]$ will all pass through one and the same point O.

We shall use barycentric coördinates $a', a'', \ldots . a^{[n+1]}$ with reference to the given n-point, so that

$$\Sigma a^{[i]} = a' + a'' + \ldots . + a^{[n+1]} = K.$$

These coördinates are the generalization of triangular coördinates in the plane, in which case K is the area of the triangle of reference, and of tetrahedral coördinates in space of three dimensions, where K is the volume of the fundamental tetrahedron. For the problem in hand it is important to regard the absolute values of these coördinates, and not, as in projective geometry, merely their ratios.

The equation of the region at infinity, a flat R_{n-1}, is then

$$\Sigma a^{[i]} = 0;$$

and, if we denote by a_{ik} the length of the edge A_iA_k, the equation of the spherical S_{n-1} circumscribing the fundamental $(n+1)$-point $A_1A_2 \ldots . A_{n+1}$ will be

$$\Omega = \sum_{i=1}^{n} \sum_{k=i+1}^{n+1} a^2{}_{ik} a^{[i]} a^{[k]} = 0,$$

while the equation of *any* spherical S_{n-1} will be of the form

$$\Sigma \lambda^{[i]} a^{[i]} . \Sigma a^{[i]} - \Omega = 0,$$

where the $\lambda^{[i]}$ are constant coefficients.

Now the coördinates of any vertex A_i of the fundamental n-point are of course all zero except $a_i{}^{[i]}$, which is equal to K; and the coördinates of any of the intermediate points A_{ik} are all zero except two, $a_{ik}{}^{[i]}$ and $a_{ik}{}^{[k]}$, whose sum is equal to K. If for brevity we write

$$\omega_i = \sum_{k=1}^{n+1} a_{ik}{}^2 a_{ik}{}^{[i]} a^{[k]}, \ [k \gtrless i]$$

the equation of the spherical S_{n-1} through the points $[A_i;\ A_{i1}, A_{i2}, \ldots . A_{i,\,n+1}]$ is readily found to be

$$\omega_i \Sigma a^{[i]} - K\Omega = 0.$$

We see however that

$$K\Omega = \Sigma \omega_i a^{[i]}$$

identically, and hence that *every one of the spherical S_{n-1} in question passes through the point O for which*

$$\omega_1 = \omega_2 = \omega_3 = \ldots\ldots = \omega_{n+1},$$

and the theorem is proved.

We proceed to the discussion of some special cases.

I. If the points A_{ik} are the middle points of the edges, it is evident that

$$\omega_i = \frac{K}{2} \sum_{k=1}^{n+1} a_{ik}{}^2 a^{[k]} = \frac{K}{2} \frac{\partial \Omega}{\partial a^{[i]}},$$

and the point O is the center of the spherical S_{n-1} circumscribed about the fundamental n-point.

II. If the flat spaces

$$\omega_1 = 0,\ \omega_2 = 0,\ \ldots\ldots \omega_{n+1} = 0$$

have a point in common, the point O will be that point and it will then lie on the circumscribing S_{n-1}.

Now this will be the case, for example, if, when n is *even*, the points A_{ik} are the intersections with the edges of a flat R_{n-1}. For, let the equation of this R_{n-1} be

$$\lambda_1 a' + \lambda_2 a'' + \ldots\ldots + \lambda_{n+1} a^{[n+1]} = 0.$$

The coördinates of A_{ik} will then be

$$a_{ik}{}^{[i]} = \frac{\lambda_k . K}{\lambda_k - \lambda_i},\quad a_{ik}{}^{[k]} = \frac{\lambda_i . K}{\lambda_i - \lambda_k}$$

and the resultant of the equations $\omega_1 = 0$ will be the determinant

$$\begin{vmatrix} 0 & \dfrac{a_{12}{}^2 \lambda_2 K}{\lambda_2 - \lambda_1} & \dfrac{a_{13}{}^2 \lambda_3 K}{\lambda_3 - \lambda_1} \cdots\cdots\cdots & \dfrac{a^2{}_{1,\, n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_1} \\ \dfrac{a_{21}{}^2 \lambda_1 K}{\lambda_1 - \lambda_2} & 0 & \dfrac{a_{23}{}^2 \lambda_3 K}{\lambda_3 - \lambda_2} \cdots\cdots\cdots & \dfrac{a^2{}_{2,\, n+1} \lambda_{n+2} K}{\lambda_{n+1} - \lambda_2} \\ \dfrac{a_{31}{}^2 \lambda_1 K}{\lambda_1 - \lambda_3} & \dfrac{a_{32}{}^2 \lambda_2 K}{\lambda_2 - \lambda_3} & 0 \quad \cdots\cdots\cdots & \dfrac{a^2{}_{3,\, n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_3} \\ \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ \dfrac{a^2{}_{n+1,\, 1} \lambda_1 K}{\lambda_1 - \lambda_{n+1}} & \dfrac{a^2{}_{n-1,\, 2} \lambda_2 K}{\lambda_2 - \lambda_{n+1}} & \dfrac{a^2{}_{n-1,\, 3} \lambda_3 K}{\lambda_3 - \lambda_{n+1}} \cdots\cdots\cdots & 0 \end{vmatrix}$$

If we now divide the columns of this determinant respectively by λ_1, λ_2, λ_3, ...λ_{n+1}, the quotient is a zero-axial skew determinant, of *odd* order when n is *even* and therefore vanishing. In this case, then, the flat spaces $\omega_1=0$, $\omega_2=0$, $\omega_3=0$, ...$\omega_{n+1}=0$ have a point in common; this will be the point O and it lies on the circumscribing S_{n-1}. In particular, for the case of a plane triangle ($n=2$), if the points A_{23}, A_{31}, A_{12} are collinear, the point O lies on the circumscribing circle,—a known theorem.

If, however, n is odd, the above determinant does not vanish. In this case the point O is a point of the intersecting R_{n-1}. For, since O is determined by

$$\omega_1=\omega_2=\omega_3=\ldots.=\omega_{n+1},$$

we may write

$$\omega_1-\rho=0,\ \omega_2-\rho=0,\ \ldots.\omega_{n+1}-\rho=0,$$

and the resultant of these $n+1$ equations and of the equation of the given R_{n-1} is the determinant

$$\begin{vmatrix} 0 & \dfrac{a_{12}^2\lambda_2 K}{\lambda_2-\lambda_1} & \dfrac{a_{13}^2\lambda_3 K}{\lambda_3-\lambda_1} & \cdots\cdots & \dfrac{a^2_{1,\,n+1}\lambda_{n+1}K}{\lambda_{n+1}-\lambda_1} & -1 \\ \dfrac{a_{21}^2\lambda_1 K}{\lambda_1-\lambda_2} & 0 & \dfrac{a_{23}^2\lambda_3 K}{\lambda_3-\lambda_2} & \cdots\cdots & \dfrac{a^2_{2,\,n+1}\lambda_{n+1}K}{\lambda_{n+1}-\lambda_2} & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{a^2_{n+1,\,1}\lambda_1 K}{\lambda_1-\lambda_{n+1}} & \dfrac{a^2_{n+1,\,2}\lambda_2 K}{\lambda_2-\lambda_{n+1}} & \dfrac{a^2_{n+1,\,3}\lambda_3 K}{\lambda_3-\lambda_{n+1}} & \cdots\cdots & 0 & -1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots\cdots & \lambda_{n+1} & 0 \end{vmatrix}$$

and, if we divide the first $n+1$ columns of this determinant by λ_1, λ_2, λ_3, ...λ_{n+1}, respectively, the quotient is a zero-axial skew determinant, which is of odd order if n is *odd*, and will then vanish.

In particular, for the case of a tetrahedron ($n=3$), if the points A_{ik} are coplanar, the point O lies on the same plane, and we have a correspondence between points and planes, in which to every plane is coördinated a point lying in that plane. The relation is not, however, uniquely reversible, as in the case of the ordinary null-system, for to every point O correspond *six* planes.

Finally, if we consider the S_{n-1} in sets of n, each such set will have a second point of intersection in addition to the point O. These points will be situated in the respective faces of the n-point,—meaning by faces the flat R_{n-1} determined by the vertices taken n at a time. It is evident that, if n is odd, these points will lie on the circumscribing S_{n-1}; while if n is even, they will lie on the the R_{n-1} whose intersections with the edges determine the points A_{ik}.

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